ANNOUNCEMENTS

- Homework 0 grading (close to done)
- Homework 1 posted; due on Monday Sep 10, 11:59 PM
- Start early! (Post comments/questions on canvas...)
  - Unless immediate, argue why algorithm is correct
  - Three steps – algorithm description, proof of correctness, complexity analysis
LAST CLASS

- Divide and conquer
  - Break instance into smaller pieces, solve pieces *separately*, combine solutions
  - Instances disjoint in examples we saw – not necessary (today)
- Merge-sort algorithm

- Recurrences
  - Guess and prove by induction
  - Plug and chug
  - Recurrence tree

- Recurrences
  - Master theorem: link on homepage
  - Akra–Bazzi Theorem
DIVIDE AND CONQUER

PROBLEM INSTANCE

Divide step

Solve sub-problems

Combine/“conquer”

FULL SOLUTION
TODAY'S PLAN

▸ More examples – divide and conquer
  ▶ Continue integer multiplication
  ▶ Median finding

▸ Introduction – dynamic programming
EXAMPLE 2: MULTIPLYING INTEGERS

Question: given two $n$ digit numbers, how efficiently can we multiply them?

- Elementary school algorithm: takes time $O(n^2)$
- Basic divide and conquer – recap
Some observations:

\[ A = 10^{n/2}X + Y \]
\[ B = 10^{n/2}Z + W \]

\[ AB = 10^n \overline{XZ} + 10^{n/2} \overline{YZ} + XW + YW \rightarrow \text{adding 4 \#s, each with } \approx 2n \text{ digits} \]

- Divide and conquer alg: compute \( XZ, YZ, XW, YW \), compute sum above

- Gives recurrence: \( T(n) = 4T(n/2) + O(n) \)
SOLVING RECURRENCE

\[ T(n) = 4T\left(\frac{n}{2}\right) + O(n) \]

\[ T(n) = O(n^2) \]

Can we do better than elementary school Alg?
CAN WE DO BETTER?

Reason to hope: we only need to find three quantities:

\( XZ, (XW + YZ), YW \)
Can we do better?

**Reason to hope:** we only need to find three quantities:

\[ XZ, (XW + YZ), YW \]

\[ \checkmark \quad \checkmark \quad \checkmark \]

**Kartsuba’s observation:**

\[ XW + YZ = (X + Y)(W + Z) - XZ - YW \]

We need only THREE multiplications!

\[ T(n) = 3 \cdot T\left(\frac{n}{2}\right) + O(n) \]
OVERALL RUNNING TIME

\[ T(n) = 3T\left(\frac{n}{2}\right) + c \cdot n \]

\[ = C \cdot n + 3 \left[ C \cdot \frac{n}{2} + 3T\left(\frac{n}{4}\right) \right] \]

\[ = C \cdot n + \left(\frac{3}{2}\right) \cdot Cn + 3^2 \cdot T\left(\frac{n}{4}\right) \]

\[ = \ldots \]

\[ = Cn \left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \ldots\right) \]

\[ = Cn \cdot \left(\frac{3}{2}\right)^r + \frac{\log_2 n}{3} \log_2 n + 3^r T\left(\frac{n}{2^r}\right) \approx n \]

Base case:
\[ \frac{m}{2^r} = 1 \Leftrightarrow r = \log_2 n \]
EVEN BETTER?

- Can we do linear time?
- HW1 – will see something approaching linear

- Fast Fourier Transform: \( \sim O(n \log \log n) \) time alg.

- Open problem if we can do better.
COMPUTING THE MEDIAN

**Question:** given an unsorted list $A[0], A[1], \ldots, A[n-1]$, find the $(n/2)$th smallest element

- Sort the array, output $\frac{n}{2}$th smallest $\Rightarrow O(n \log n)$

**Divide and conquer?**
Attempt:

\[ \frac{m}{2} \quad \frac{n}{2} \]

...medians of the pieces don't give much meaningful information.

\[ :- ( \]
KEY INSIGHT: “APPROXIMATE MEDIAN” HELPS!

\[ A[0], A[\phi], \ldots, A[m-1] \]


middle \( \frac{n}{2} \) elements

\[ A[1], A[2], A[0], \ldots \}
\{ x \}

\[ \frac{2n}{3}, \frac{m}{3} \]
Observations:

1) We can now restrict search to a $\frac{2n}{3}$ sized array ($\ll n$)

2) Need to be careful: answer is not median of sub-array, but $\frac{n}{2}$ smallest element.

Change our goal to solving a more general problem:

Given an array & some $k$, find the $k^{th}$ smallest element. $\left[\text{Select}(A, k)\right]$

$1 \& 2 \Rightarrow \text{Select } (A, \frac{n}{2}) = \text{Select } \left[ \frac{B}{1}, \frac{n}{2} \right]$

All elements $\leq x$

$T(n) = T\left( \frac{2n}{3} \right) + O(n) \rightarrow$ time taken to find sub-array $B,$

$= Cn + C \left( \frac{2n}{3} \right) + C \left( \frac{2}{3} \right)^2 n + \ldots$ $B,$

$= Cn \left( 1 + \frac{2}{3} + \left( \frac{2}{3} \right)^2 + \left( \frac{2}{3} \right)^3 + \ldots \right) = c'n.$
Let $x'$ be the $\frac{n+1}{3}$ smallest element of $B$.

The $\frac{n}{2}$ smallest element of $B$ is the $\left(\frac{n}{2} - \frac{n}{3}\right)$th smallest element of $B'$.

$= \text{Select} \left( B', \frac{n}{2} - \frac{n}{3} \right)$

Overall idea: Select $(A, k)$ will link to Select $(B, k')$ on a new array $B$ s.t. $|B| < \frac{3}{4} |A|$.

(a) Find approximate median

(b) Will make recursive call on a new array.
MEDIAN OF MEDIANS IS APPROXIMATE MEDIAN!

\[ T(n) = T\left(\frac{n}{10}\right) + O(n) + T\left(\frac{2n}{3}\right) \]
DIVIDE AND CONQUER — OVERVIEW

- Problem “cleanly” divides into not-too-many sub-problems
- Solutions easily (efficiently) combined
- Leads to simple, efficient solutions
- Analysis by induction, complexity by recurrences
OVERLAP IN RECURSIVE CALLS

▸ Divide & conquer: usually sub-problems don’t interact “much”

▸ What if they do? Can run into wastefulness – solving same sub-problem many times...

▸ Motivation for next topic – *dynamic programming*
NEW PROBLEM: SUBSET SUM

Problem: given a set of non-negative integers $A[0], A[1], ..., A[n-1]$, and an integer $S$, find if there is a subset of $A[i]$ whose sum is $S$.

Brute-force:
- Try all subsets: find if any sum to $S$.


$(2^n \cdot n)$ - time
Try 1: divide into two halves

\[ T(n) = (S + 1) T\left( \frac{n}{2} \right) \]

\[ \log_2 S = (S + 1) \log_2 n \]

= \( n \)
RUNNING TIME

\[ \log_2 n \]
Problem: given a set of non-negative integers $A[0], A[1], \ldots, A[n-1]$, and an integer $S$, find if there is a subset of $A[i]$ whose sum is $S$.

- See if first element is picked or not

\[
\text{Look for a sum of } S - A[0] \text{ in } A[1], \ldots, A[n-1]
\]

\[
\text{Look for a sum of a sum of } A[0] \text{ in } A[1], \ldots, A[n-1]
\]

\[
\text{Takes time } 2^n
\]
Problem: given a set of non-negative integers $A[0], A[1], \ldots, A[n-1]$, and an integer $S$, find if there is a subset of $A[i]$ whose sum is $S$.

- See if first element is picked or not
- Is this really divide and conquer?
RUNNING TIME
LOOKING MORE CLOSELY — EXAMPLE

\[ A[] = \{1, 2, 3, 5, 7, 9, 10, 11\}; \ S = 20 \]
AVOIDING MULTIPLE SOLVES ON “SAME” INSTANCE

▶ Store answers!

▶ How much are we storing?

▶ Running time